Module 5: Two Dimensional Problems in Cartesian Coordinate System

5.3.1 SOLUTIONS OF TWO-DIMENSIONAL PROBLEMS BY THE USE OF POLYNOMIALS

The equation given by

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \quad (5.13)$$

will be satisfied by expressing Airy's function $\phi(x, y)$ in the form of homogeneous polynomials.

(a) Polynomial of the First Degree

Let $\phi_1 = a_1 x + b_1 y$

Now, the corresponding *stresses* are

$$\sigma_x = \frac{\partial^2\phi_1}{\partial y^2} = 0$$

$$\sigma_y = \frac{\partial^2\phi_1}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2\phi_1}{\partial x\partial y} = 0$$

Therefore, this stress function gives a stress free body.

(b) Polynomial of the Second Degree

Let $\phi_2 = \frac{a_2}{2}x^2 + b_2 xy + \frac{c_2}{2}y^2$

The corresponding stresses are

$$\sigma_x = \frac{\partial^2\phi_2}{\partial y^2} = c_2$$

$$\sigma_y = \frac{\partial^2\phi_2}{\partial x^2} = a_2$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -b_2$$

This shows that the above stress components do not depend upon the co-ordinates x and y, i.e., they are constant throughout the body representing a constant stress field. Thus, the stress function ϕ_2 represents a state of uniform tensions (or compressions) in two perpendicular directions accompanied with uniform shear, as shown in the Figure 5.3 below.

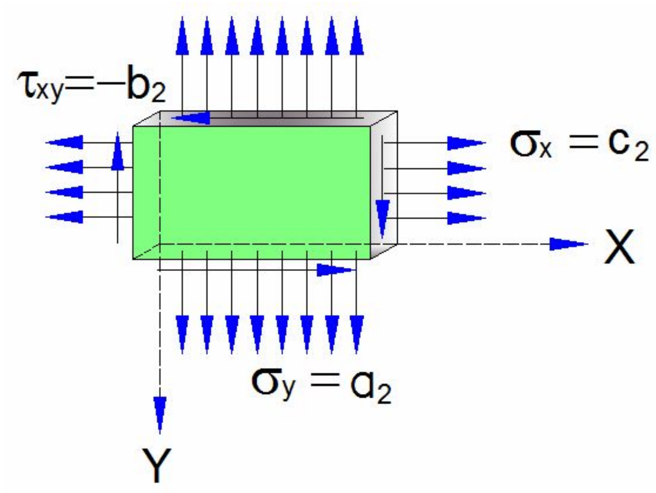


Figure 5.3 Constant Stress field

(c) Polynomial of the Third Degree

Let $\phi_3 = \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2 y + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3$

The corresponding stresses are

$$\sigma_x = \frac{\partial^2 \phi_3}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 \phi_3}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x \partial y} = -b_3 x - c_3 y$$

This stress function gives a linearly varying stress field. It should be noted that the magnitudes of the coefficients a_3, b_3, c_3 and d_3 are chosen freely since the expression for ϕ_3 is satisfied irrespective of values of these coefficients.

Now, if $a_3 = b_3 = c_3 = 0$ except d_3, we get from the stress components

$\sigma_x = d_3 y$

$\sigma_y = 0$ and $\tau_{xy} = 0$

This corresponds to pure bending on the face perpendicular to the *x*-axis.

$\therefore$ At $y = -h$, $\sigma_x = -d_3 h$

and At $y = +h,\ \sigma_x = +d_3 h$

The variation of σ_x with y is linear as shown in the Figure 5.4.

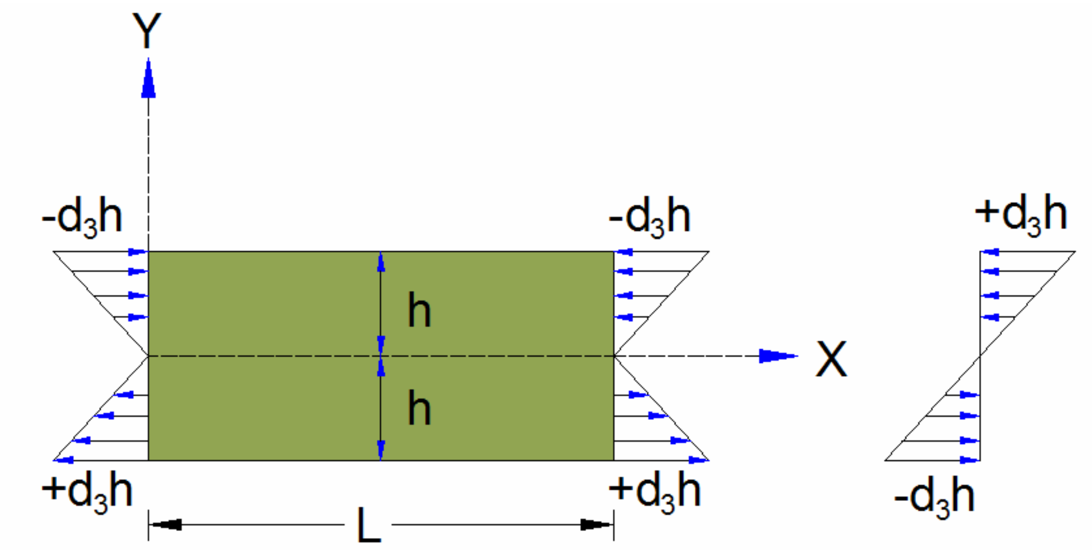


Figure 5.4 Variation of Stresses

Similarly, if all the coefficients except b_3 are zero, then we get

$\sigma_x = 0$

$\sigma_y = b_3 y$

$\tau_{xy} = -b_3 x$

The stresses represented by the above stress field will vary as shown in the Figure 5.5.

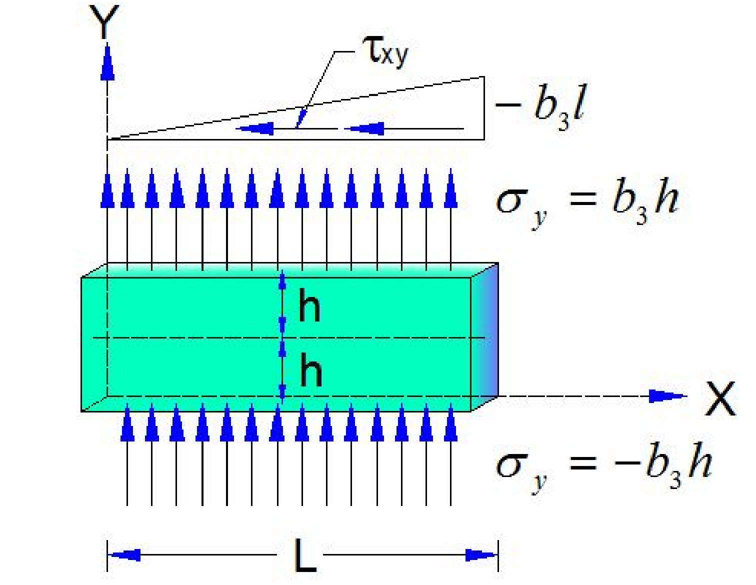


Figure 5.5 Variation of Stresses

In the Figure 5.5, the stress σ_y is constant with x (i.e. constant along the span L of the beam), but varies with y at a particular section. At $y = +h$, $\sigma_y = b_3 h$ (i.e., tensile), while at $y = -h$, $\sigma_y = -b_3 h$ (i.e. compressive). σ_x is zero throughout. Shear stress τ_{xy} is zero at $x = 0$ and is equal to $-b_3 L$ at $x = L$. At any other section, the shear stress is proportional to x.

(d) Polynomial of the Fourth Degree

Let $\phi_4 = \frac{a_4}{12} x^4 + \frac{b_4}{6} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{6} x y^3 + \frac{e_4}{12} y^4$

The corresponding stresses are given by

$$\sigma_x = c_4 x^2 + d_4 xy + e_4 y^2$$

$$\sigma_y = a_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = -\left(\frac{b_4}{2}\right) x^2 - 2c_4 xy - \left(\frac{d_4}{2}\right) y^2$$

Now, taking all coefficients except d_4 equal to zero, we find

$$\sigma_x = d_4 xy, \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{d_4}{2} y^2$$

Assuming d_4 positive, the forces acting on the beam are shown in the Figure 5.6.

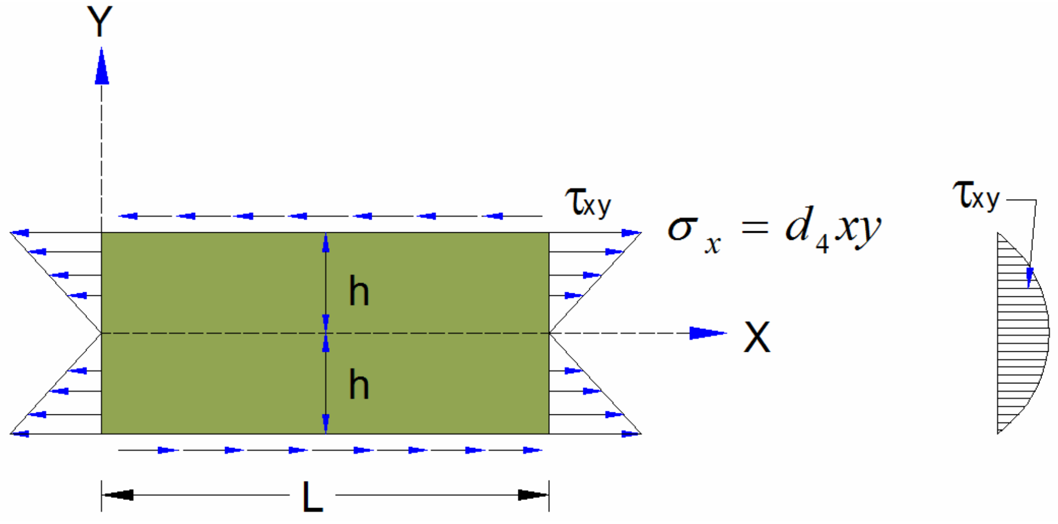


Figure 5.6 Stresses acting on the beam

On the longitudinal sides, $y = \pm h$ are uniformly distributed shearing forces. At the ends, the shearing forces are distributed according to a parabolic distribution. The shearing forces acting on the boundary of the beam reduce to the couple.

Therefore, $M = \frac{d_4 h^2 L}{2} 2h - \frac{1}{3}\frac{d_4 h^2}{2} 2hL$

Or $M = \frac{2}{3} d_4 h^3 L$

This couple balances the couple produced by the normal forces along the side $x = L$ of the beam.

(e) Polynomial of the Fifth Degree

$$Let\, \varphi_5 = \frac{a_5}{20}x^5 + \frac{b_5}{12}x^4 y + \frac{c_5}{6}x^3 y^2 + \frac{d_5}{6}x^2 y^3 + \frac{e_5}{12}xy^4 + \frac{f_5}{20}y^5$$

The corresponding stress components are given by

$$\sigma_x = \frac{\partial^2 \phi_5}{\partial y^2} = \frac{c_5}{3}x^3 + d_5 x^2 y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3$$

$$\sigma_y = \frac{\partial^2 \phi_5}{\partial x^2} = a_5 x^3 + b_5 x^2 y + c_5 xy^2 + \frac{d_5}{3}y^3$$

$$\tau_{xy} = -\frac{\partial^2 \phi_5}{\partial x \partial y} = -\frac{1}{3}b_5 x^3 - c_5 x^2 y - d_5 xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3$$

Here the coefficients a_5, b_5, c_5, d_5 are arbitrary, and in adjusting them we obtain solutions for various loading conditions of the beam.

Now, if all coefficients, except d_5, equal to zero, we find

$$\sigma_x = d_5\left(x^2 y - \frac{2}{3}y^3\right)$$

$$\sigma_y = \frac{1}{3}d_5 y^3$$

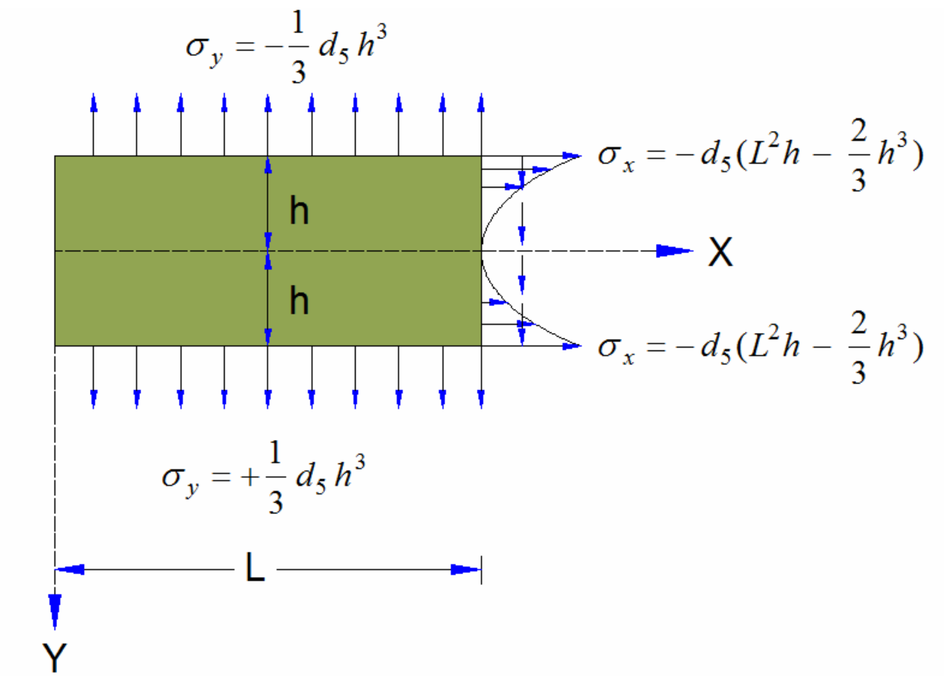
$$\tau_{xy} = -d_5 xy^2$$

Case (i)

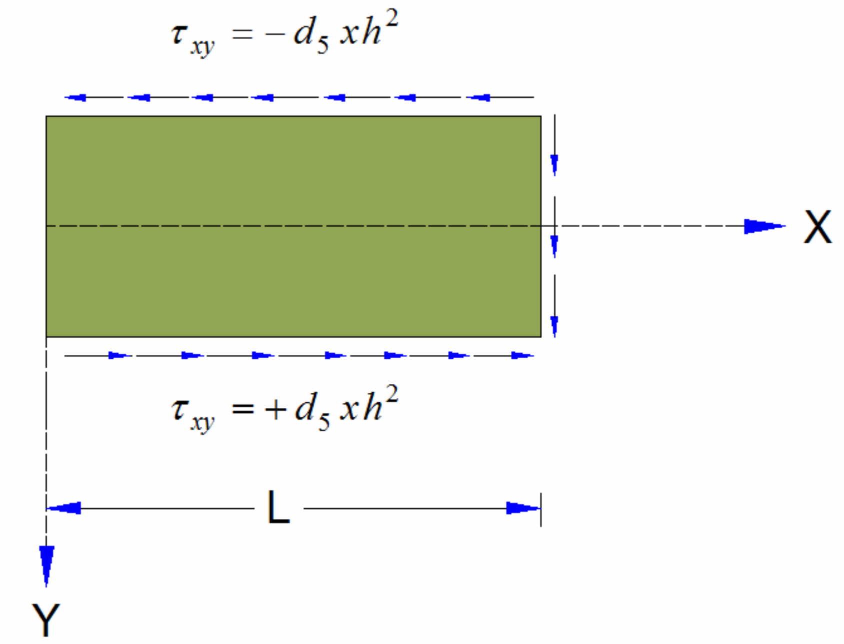
The normal forces are uniformly distributed along the longitudinal sides of the beam.

Case (ii)

Along the side $x = L$, the normal forces consist of two parts, one following a linear law and the other following the law of a cubic parabola. The shearing forces are proportional to x on the longitudinal sides of the beam and follow a parabolic law along the side $x = L$. The distribution of the stresses for the **Case (i)** and **Case (ii)** are shown in the Figure 5.7.



Case (i)



Case (ii)

Figure 5.7 Distribution of forces on the beam

5.3.2 BENDING OF A NARROW CANTILEVER BEAM SUBJECTED TO END LOAD

Consider a cantilever beam of narrow rectangular cross-section carrying a load P at the end as shown in Figure 5.8.

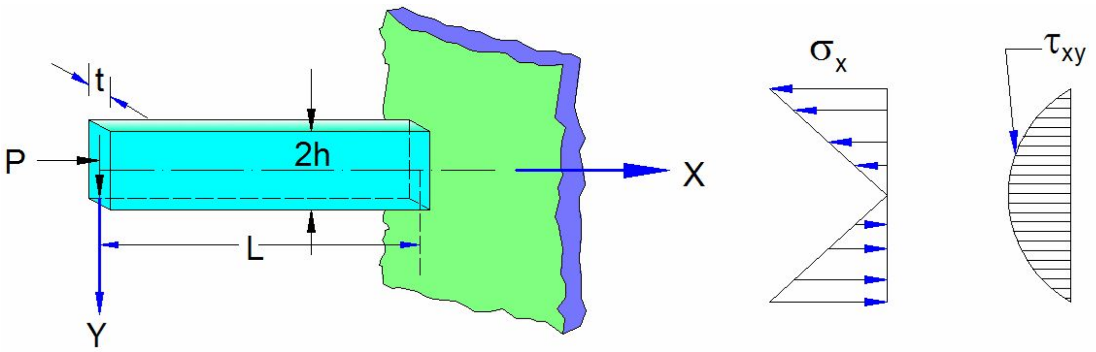


Figure 5.8 Cantilever subjected to an end load

The above problems may be considered as a case of plane stress provided that the thickness of the beam t is small relative to the depth $2h$.

Boundary Conditions

$$(\tau_{xy})_{At\ y=\pm h} = 0$$
$$(\sigma_y)_{At\ y=\pm h} = 0 \tag{5.14}$$

These conditions express the fact that the top and bottom edges of the beam are not loaded. Further, the applied load P must be equal to the resultant of the shearing forces distributed across the free end.

Therefore, $P = -\int_{-h}^{+h} \tau_{xy}\ 2b\ dy$ (5.14a)

By Inverse Method

As the bending moment varies linearly with x, and σ_x at any section depends upon y, it is reasonable to assume a general expression of the form

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = c_1 xy \tag{5.14b}$$

where c_1 = constant. Integrating the above twice with respect to y, we get

$$\phi = \frac{1}{6} c_1 xy^3 + y f_1(x) + f_2(x) \tag{5.14c}$$

where $f_1(x)$ and $f_2(x)$ are functions of x to be determined. Introducing the ϕ thus obtained into Equation (5.12), we have

$$y\frac{d^4 f_1}{dx^4} + \frac{d^4 f_2}{dx^4} = 0 \tag{5.14d}$$

Since the second term is independent of y, there exists a solution for all x and y provided that

$$\frac{d^4 f_1}{dx^4} = 0 \text{ and } \frac{d^4 f_2}{dx^4} = 0$$

Integrating the above, we get

$$f_1(x) = c_2x^3 + c_3x^2 + c_4x + c_5$$

$$f_2(x) = c_6x^3 + c_7x^2 + c_8x + c_9$$

where c_2, c_3……., c_9 are constants of integration.

Therefore, (5.14c) becomes

$$\phi = \frac{1}{6} c_1 xy^3 + (c_2x^3 + c_3x^2 + c_4x + c_5)y + c_6x^3 + c_7x^2 + c_8x + c_9$$

Now, by definition,

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 6(c_2 y + c_6)x + 2(c_3 y + c_7)$$

$$\tau_{xy} = -\left(\frac{\partial^2\phi}{\partial x\partial y}\right) = -\frac{1}{2}c_1y^2 - 3c_2x^2 - 2c_3x - c_4 \qquad (5.14e)$$

Now, applying boundary conditions to (5.14e), we get

$c_2 = c_3 = c_6 = c_7 = 0$ and $c_4 = -\frac{1}{2}c_1h^2$

Also, $-\int_{-h}^{+h}\tau_{xy}\, 2b\, dy = \int_{-h}^{h}\frac{1}{2}c_1\, 2b\,(y^2 - h^2)dy = P$

Solving, $c_1 = -\left(\frac{3P}{4b\,h^3}\right) = -\left(\frac{P}{I}\right)$

where $I = \frac{4}{3}bh^3$ is the moment of inertia of the cross-section about the neutral axis.

From Equations (5.14b) and (5.14e), together with the values of constants, the stresses are found to be

$$\sigma_x = -\left(\frac{Pxy}{I}\right),\ \sigma_y = 0,\ \tau_{xy} = \frac{-P}{2I}(h^2 - y^2)$$

The distribution of these stresses at sections away from the ends is shown in Figure 5.8 b

By Semi-Inverse Method

Beginning with bending moment $M_z = Px$, we may assume a stress field similar to the case of pure bending:

$$\sigma_x = -\left(\frac{Px}{I}\right)y$$

$$\tau_{xy} = \tau_{xy}(x, y) \qquad (5.14f)$$

$$\sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

The equations of compatibility are satisfied by these equations. On the basis of equation (5.14f), the equations of equilibrium lead to

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0,\ \frac{\partial\tau_{xy}}{\partial x} = 0 \qquad (5.14g)$$

From the second expression above, τ_{xy} depends only upon y. The first equation of (5.14g) together with equation (5.14f) gives

$$\frac{d\tau_{xy}}{dy} = \frac{Py}{I}$$

or $\tau_{xy} = \dfrac{Py^2}{2I} + c$

Here c is determined on the basis of $(\tau_{xy})_{y=\pm h} = 0$

Therefore, $c = -\dfrac{Ph^2}{2I}$

Hence, $\tau_{xy} = \dfrac{Py^2}{2I} - \dfrac{Ph^2}{2I}$

Or $\tau_{xy} = -\dfrac{P}{2I}(h^2 - y^2)$

The above expression satisfies equation (5.14a) and is identical with the result previously obtained.

5.3.3 PURE BENDING OF A BEAM

Consider a rectangular beam, length L, width 2b, depth 2h, subjected to a pure couple M along its length as shown in the Figure 5.9

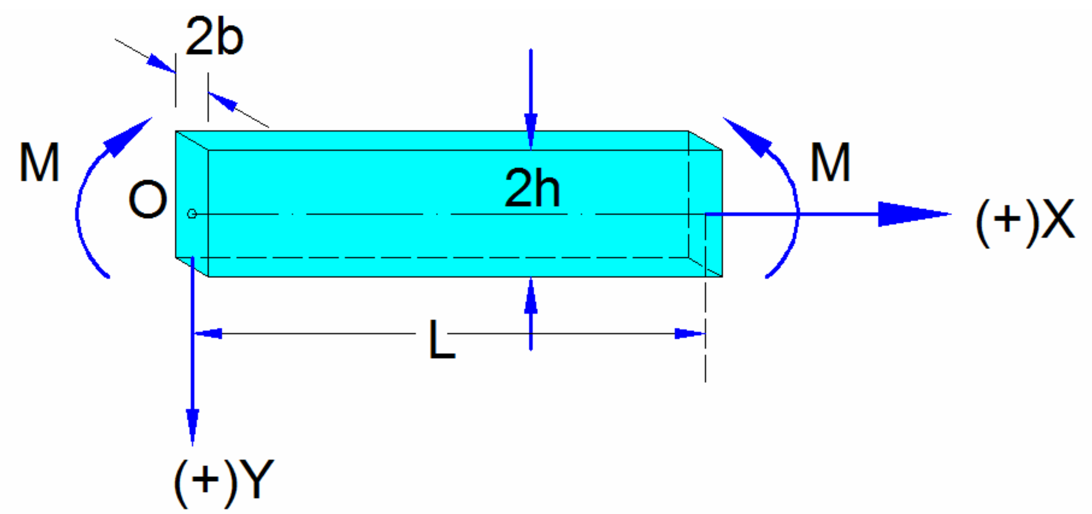


Figure 5.9 Beam under pure bending

Consider a second order polynomial such that its any term gives only a constant state of stress. Therefore

$\phi = a_2 \dfrac{x^2}{2} + b_2 xy + \dfrac{c_2 y^2}{2}$

By definition,

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2},\ \sigma_y = \frac{\partial^2\phi}{\partial x^2},\ \tau_{xy} = -\left(\frac{\partial^2\phi}{\partial x\partial y}\right)$$

$\therefore$ Differentiating the function, we get

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = c_2, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2} = a_2 \text{ and } \tau_{xy} = -\left(\frac{\partial^2\phi}{\partial x\partial y}\right) = -b_2$$

Considering the plane stress case,

$\sigma_z = \tau_{xz} = \tau_{yz} = 0$

Boundary Conditions

(a) At $y = \pm h, \sigma_y = 0$

(b) At $y = \pm h,\ \tau_{xy} = 0$

(c) At x = any value,

$2b \int_{-a}^{+a} \sigma_x y dy$ = bending moment = constant

$$\therefore 2bx \int_{-h}^{+h} c_2 y dy = 2bc_2 x\left[\frac{y^2}{2}\right]_{-h}^{+h} = 0$$

Therefore, this clearly does not fit the problem of pure bending.

Now, consider a third-order equation

$$\phi = \frac{a_3x^3}{6} + \frac{b_3}{2}x^2y + \frac{c_3xy^2}{2} + \frac{d_3y^3}{6}$$

Now, $\sigma_x = \dfrac{\partial^2\phi}{\partial y^2} = c_3x + d_3y$ (a)

$\sigma_y = a_3x + b_3y$ (b)

$\tau_{xy} = -b_3x - c_3y$ (c)

From (b) and boundary condition (a) above,

$0 = a_3x \pm b_3a$ for any value of x

$\therefore a_3 = b_3 = 0$

From (c) and the above boundary condition (b),

$0 = -b_3x \pm c_3a$ for any value of x

therefore $c_3 = 0$

hence, $\sigma_x = d_3y$

$\sigma_y = 0$

$\tau_{xy} = 0$

Obviously, Biharmonic equation is also satisfied.

i.e., $\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$

Now, bending moment = $M = 2b\int_{-h}^{+h}\sigma_x y\,dy$

i.e. $M = 2b\int_{-h}^{+h} d_3 y^2 dy$

$= 2bd_3 \int_{-h}^{+h} y^2 dy$

$= 2bd_3 \left[\frac{y^3}{3}\right]_{-h}^{+h}$

$M = 4bd_3 \frac{h^3}{3}$

Or $d3 = \frac{3M}{4bh^3}$

$d_3 = \frac{M}{I}$ where $I = \frac{4h^3 b}{3}$

Therefore, $\sigma_x = \frac{M}{I}y$

5.3.4 BENDING OF A SIMPLY SUPPORTED BEAM BY A DISTRIBUTED LOADING (UDL)

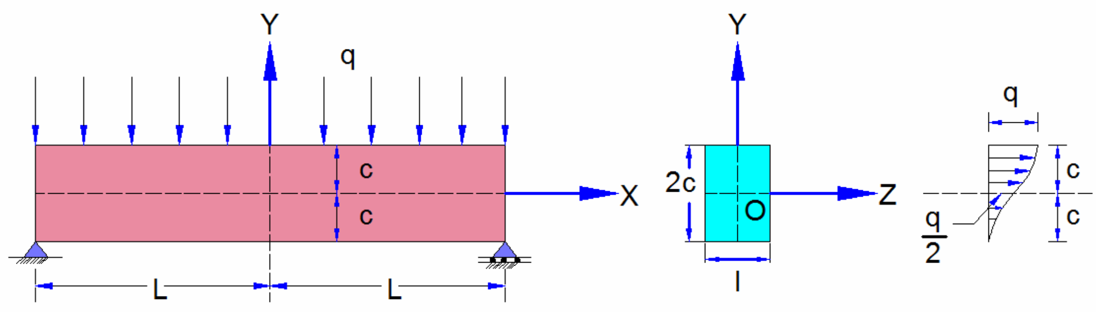


Figure 5.10 Beam subjected to Uniform load

Consider a beam of rectangular cross-section having unit width, supported at the ends and subjected to a uniformly distributed load of intensity q as shown in the Figure 5.10.

It is to be noted that the bending moment is maximum at position $x = 0$ and decreases with change in x in either positive or negative directions. This is possible only if the stress function contains even functions of x. Also, it should be noted that σ_y various from zero at $y = -c$ to a maximum value of $-q$ at $y = +c$. Hence the stress function must contain odd functions of y.

Now, consider a polynomial of second degree with

$$b_2 = c_2 = 0$$

$$\therefore \phi_2 = \frac{a_2}{2} x^2$$

a polynomial of third degree with $a_3 = c_3 = 0$

$$\therefore \phi_3 = \frac{b_3}{2} x^2 y + \frac{d_3}{6} y^3$$

and a polynomial of fifth degree with $a_5 = b_5 = c_5 = e_5 = 0$

$$\therefore \phi_5 = \frac{d_5}{6} x^2 y^3 - \frac{d_5}{30} y^5 \qquad \left[\because f_5 = -\frac{2}{3} d_5\right]$$

$$\therefore \phi = \phi_2 + \phi_3 + \phi_5$$

$$\text{or } \phi = \frac{a_2}{2} x^2 + \frac{b_3}{2} x^2 y + \frac{d_3}{6} y^3 + \frac{d_5}{6} x^2 y^3 - \frac{d_5}{30} y^5 \qquad (1)$$

Now, by definition,

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = d_3 y + d_5\left(x^2 y - \frac{2}{3}y^3\right) \qquad (2)$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = a_2 + b_3 y + \frac{d_5}{3}y^3 \qquad (3)$$

$$\tau_{xy} = -b_3 x - d_5 x y^2 \qquad (4)$$

The following boundary conditions must be satisfied.

(i) $(\tau_{xy})_{y=\pm c} = 0$

(ii) $(\sigma_y)_{y=+c} = 0$

(iii) $(\sigma_y)_{y=-c} = -q$

(iv) $\int_{-c}^{+c} (\sigma_x)_{x=\pm L}\, dy = 0$

(v) $\int_{-c}^{+c} (\tau_{xy})_{x=\pm L}\, dy = \pm qL$

(vi) $\int_{-c}^{+c} (\sigma_x)_{x=\pm L}\, y dy = 0$

The first three conditions when substituted in equations (3) and (4) give

$$-b_3 - d_5 c^2 = 0$$

$$a_2 + b_3 c + \frac{d_5}{3}c^3 = 0$$

$$a_2 - b_3 c - \frac{d_5}{3}c^3 = -q$$

which gives on solving

$$a_2 = -\frac{q}{2}, \quad b_3 = \frac{3q}{4c}, \quad d_5 = -\frac{3q}{4c^3}$$

Now, from condition (vi), we have

$$\int_{-c}^{+c}\left[d_3 y + d_5\left(x^2 y - \frac{2}{3}y^3\right)\right] y dy = 0$$

Simplifying,

$$d_3 = -d_5\left(L^2 - \frac{2}{5}h^2\right)$$

$$= \frac{3q}{4h}\left(\frac{L^2}{h^2} - \frac{2}{5}\right)$$

$$\therefore \sigma_x = \frac{3q}{4h}\left(\frac{L^2}{h^2} - \frac{2}{5}\right)y - \frac{3q}{4h^3}\left(x^2 y - \frac{2}{3}y^3\right)$$

$$\sigma_y = -\left(\frac{q}{2}\right) + \frac{3q}{4h}y - \frac{q}{4h^3}y^3$$

$$\tau_{xy} = -\left(\frac{3q}{4h}\right)x + \frac{3q}{4h^3}xy^2$$

Now, $I = \frac{1 \times (2h)^3}{12} = \frac{8h^3}{12} = \frac{2}{3}h^3$

where I = Moment of inertia of the unit width beam.

$$\therefore \sigma_x = \frac{q}{2I}\left(L^2 - x^2\right)y + \frac{q}{I}\left(\frac{y^3}{3} - \frac{h^2 y}{5}\right)$$

$$\sigma_y = -\left(\frac{q}{2I}\right)\left(\frac{y^3}{3} - h^2 y + \frac{2}{3}h^3\right)$$

$$\tau_{xy} = -\left(\frac{q}{2I}\right)x\left(h^2 - y^2\right)$$

5.3.5 NUMERICAL EXAMPLES

Example 5.1

Show that for a simply supported beam, length 2*L*, depth 2*a* and unit width, loaded by a concentrated load *W* at the centre, the stress function satisfying the loading condition is $\phi = \frac{b}{6}xy^2 + cxy$ **the positive direction of *y* being upwards, and *x* = 0 at midspan.**

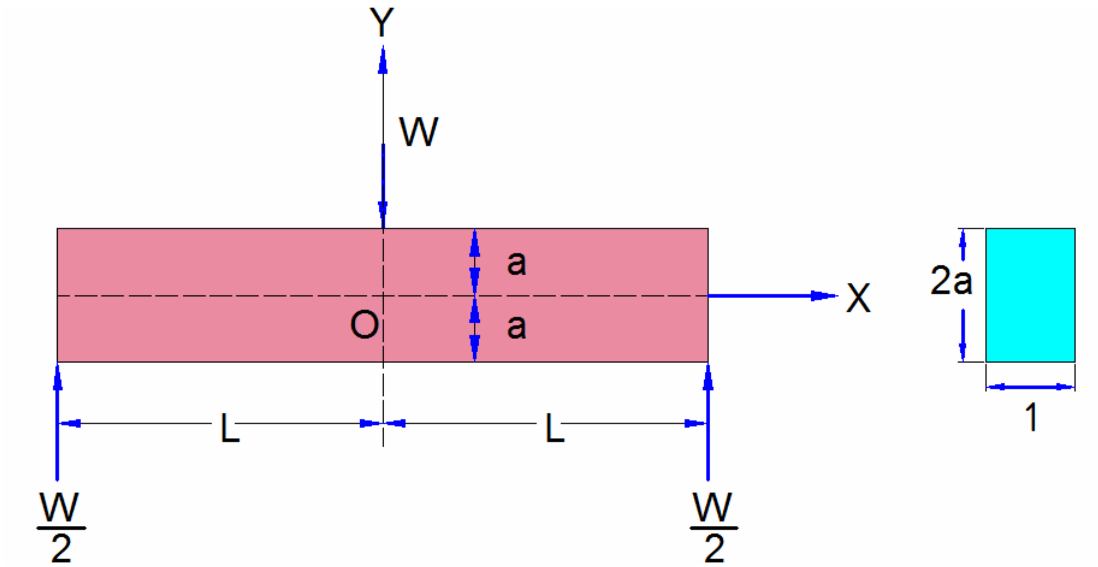


Figure 5.11 Simply supported beam

Treat the concentrated load as a shear stress suitably distributed to suit this function, and so that $\int_{-a}^{+a} \sigma_x dy = -\left(\frac{W}{2}\right)$ on each half-length of the beam. Show that the stresses are

$$\sigma_x = -\left(\frac{3}{4}\frac{W}{a^3}xy\right)$$

$$\sigma_y = 0$$

$$\tau_{xy} = -\left[\frac{3W}{8a}\left(1 - \frac{y^2}{a^2}\right)\right]$$

***Solution*:** The stress components obtained from the stress function are

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = bxy$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\left(\frac{by^2}{2}\right) + c$$

Boundary conditions are

(i) $\sigma_y = 0 \; for \; y = \pm a$

(ii) $\tau_{xy} = 0 \; for \; y = \pm a$

(iii) $-\int_{-a}^{+a} \tau_{xy} dy = \frac{W}{2} \; for \; x = \pm L$

(iv) $\int_{-a}^{+a} \sigma_x dy = 0 \; for \; x = \pm L$

(v) $\int_{-a}^{+a} \sigma_x y dy = 0 \; for \; x = \pm L$

Now,

Condition (i)

This condition is satisfied since $\sigma_y = 0$

Condition (ii)

$$0 = -\left(\frac{ba^2}{2}\right) + c$$

$$\therefore c = \frac{ba^2}{2}$$

Condition (iii)

$$\frac{W}{2} = -\int_{-a}^{+a} \frac{b}{2}\left(a^2 - y^2\right)dy$$

$$= -\frac{b}{2}\left(2a^3 - \frac{2a^3}{3}\right)$$

$$\therefore \frac{W}{2} = -\left(\frac{2a^3 b}{3}\right)$$

$$\text{or } b = -\left(\frac{3W}{4a^3}\right)$$

$$\text{and } c = -\left(\frac{3W}{8a}\right)$$

Condition (iv)

$$\int_{-a}^{+a} -\left(\frac{3W}{4a^3}\right)xydy = 0$$

Condition (v)

$$M = \int_{-a}^{+a} \sigma_x ydy$$

$$= \int_{-a}^{+a} -\left(\frac{3W}{4a^3}\right)xy^2dy$$

$$\therefore M = \frac{Wx}{2}$$

Hence stress components are

$$\sigma_x = -\left(\frac{3W}{4a^3}\right)xy$$

$$\sigma_y = 0$$

$$\tau_{xy} = \left(\frac{3W}{4a^3}\frac{y^2}{2}\right) - \left(\frac{3}{8}\frac{W}{a}\right)$$

$$\therefore \tau_{xy} = -\left[\frac{3W}{8a}\left(1 - \frac{y^2}{a^2}\right)\right]$$

Example 5.2

Given the stress function $\phi = \left(\frac{H}{\pi}\right)z\tan^{-1}\left(\frac{x}{z}\right)$. Determine whether stress function ϕ is admissible. If so determine the stresses.

***Solution*: ** For the stress function ϕ to be admissible, it has to satisfy bihormonic equation. Bihormonic equation is given by

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial z^2} + \frac{\partial^4\phi}{\partial z^4} = 0 \qquad \text{(i)}$$

Now,

$$\frac{\partial\phi}{\partial z} = \frac{H}{\pi}\left[-\left(\frac{xz}{x^2+z^2}\right) + \tan^{-1}\left(\frac{x}{z}\right)\right]$$

$$\frac{\partial^2\phi}{\partial z^2} = \left(\frac{H}{\pi}\right)\frac{1}{(x^2+z^2)^2}\left[2xz^2 - xz^2 - x^3 - xz^2 - x^3\right]$$

$$\therefore \frac{\partial^2\phi}{\partial z^2} = -\left(\frac{2H}{\pi}\right)\left[\frac{x^3}{(x^2+z^2)^2}\right]$$

Also,
$$\frac{\partial^3\phi}{\partial z^3} = \frac{H}{\pi}\left[\frac{8x^3 z}{(x^2+z^2)^3}\right]$$

$$\frac{\partial^4\phi}{\partial z^4} = \frac{H}{\pi}\left[\frac{8x^5 - 40x^3z^2}{(x^2+z^2)^4}\right]$$

$$\frac{\partial^3\phi}{\partial z^2\partial x} = -\left(\frac{2H}{\pi}\right)\left[\frac{3x^3z^2 - x^4}{(x^2+z^2)^3}\right]$$

$$\frac{\partial^4\phi}{\partial z^2\partial x^2} = \frac{H}{\pi}\left[\frac{64x^3z^2 - 24xz^4 - 8x^5}{(x^2+z^2)^4}\right]$$

Similarly,

$$\frac{\partial\phi}{\partial x} = \frac{H}{\pi}\left[\frac{z^2}{(x^2+z^2)}\right]$$

$$\frac{\partial^2\phi}{\partial x^2} = -\left(\frac{2H}{\pi}\right)\left[\frac{xz^2}{(x^2+z^2)^2}\right]$$

$$\frac{\partial^3\phi}{\partial x^3} = \frac{2H}{\pi}z^2\left[\frac{(3x^2 - z^2)}{(x^2+z^2)^3}\right]$$

$$\frac{\partial^4\phi}{\partial x^4} = \frac{H}{\pi}\left[\frac{24xz^4 - 24x^3z^2}{(x^2-z^2)^4}\right]$$

Substituting the above values in (i), we get

$$\frac{4}{\pi}\frac{1}{(x^2+z^2)^4}\left[24xz^4 - 24x^3z^2 + 64x^3z^2 - 24xz^4 - 8x^5 + 8x^5 - 40x^3z^2\right] = 0$$

Hence, the given stress function is admissible.
Therefore, the stresses are

$$\sigma_x = \frac{\partial^2\phi}{\partial z^2} = -\left(\frac{24}{\pi}\right)\left[\frac{x^3}{(x^2+z^2)^2}\right]$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = -\left(\frac{24}{\pi}\right)\left[\frac{x^2}{(x^2+z^2)^2}\right]$$

and
$$\tau_{xy} = \frac{\partial^2\phi}{\partial x\partial z} = -\left(\frac{24}{\pi}\right)\left[\frac{x^2 z}{(x^2+z^2)^2}\right]$$

Example 5.3

Given the stress function: $\phi = -\left(\frac{F}{d^3}\right)xz^2(3d - 2z)$**.**

Determine the stress components and sketch their variations in a region included in *z* = 0, *z* = *d*, *x* = 0, on the side *x* positive.

***Solution*:** The given stress function may be written as

$$\phi = -\left(\frac{3F}{d^2}\right)xz^2 + \left(\frac{2F}{d^3}\right)xz^3$$

$$\therefore \frac{\partial^2\phi}{\partial z^2} = -\left(\frac{6Fx}{d^2}\right) + \left(\frac{12F}{d^3}\right)xz$$

and $\frac{\partial^2\phi}{\partial x^2} = 0$

also $\frac{\partial^2\phi}{\partial x\partial z} = -\left(\frac{6Fz}{d^2}\right) + \left(\frac{6F}{d^3}\right)z^2$

Hence $\sigma_x = -\left(\frac{6Fx}{d^2}\right) + \left(\frac{12F}{d^3}\right)xz$ (i)

$\sigma_z = 0$ (ii)

$\tau_{xz} = -\frac{\partial^2\varphi}{\partial x\partial z} = -\left(\frac{6Fz}{d^2}\right) + \left(\frac{6F}{d^3}\right)z^2$ (iii)

VARIATION OF STRESSES AT CERTAIN BOUNDARY POINTS

(a) Variation of σ_x

From (i), it is clear that σ_x varies linearly with *x*, and at a given section it varies linearly with *z*.

$\therefore$ At *x* = 0 and *z* = $\pm d$, $\sigma_x = 0$

At *x* = *L* and z = 0, $\sigma_x = -\left(\frac{6FL}{d^2}\right)$

At *x* = *L* and z = +*d*, $\sigma_x = -\left(\frac{6FL}{d^2}\right) + \left(\frac{12F}{d^3}\right)Ld = \frac{6FL}{d^2}$

At *x* = *L* and z = -*d*, $\sigma_x = -\left(\frac{6FL}{d^2}\right) - \left(\frac{12F}{d^3}\right)Ld = -\left(\frac{18FL}{d^2}\right)$

The variation of σ_x is shown in the figure below

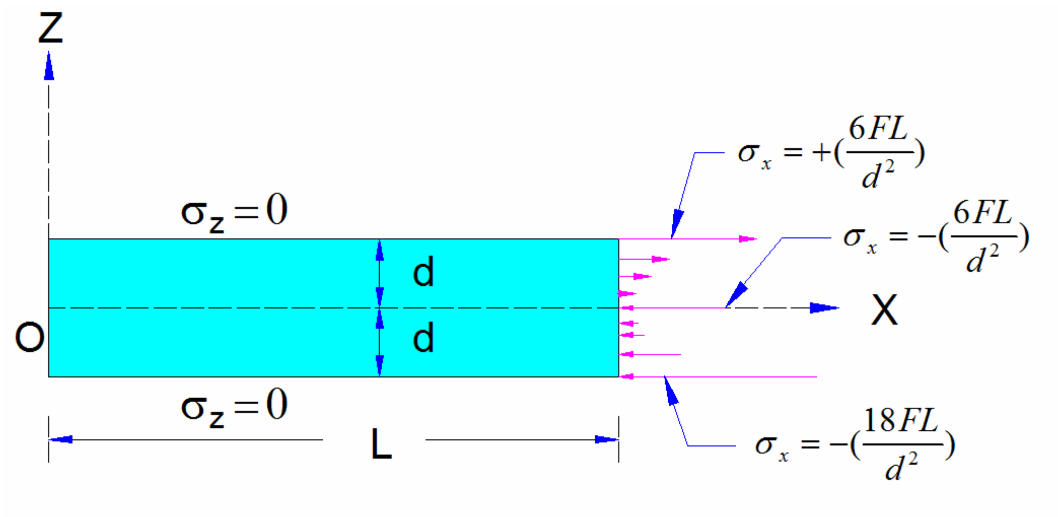


Figure 5.12 Variation of σ_x

(b) Variation of σ_z

σ_z is zero for all values of x.

(c) Variation of τ_{xz}

We have $\tau_{xz} = \left(\frac{6Fz}{d^2}\right) - \left(\frac{6F}{d^3}\right).z^2$

From the above expression, it is clear that the variation of τ_{xz} is parabolic with z. However, τ_{xz} is independent of x and is thus constant along the length, corresponding to a given value of z.

$\therefore$ At z = 0, $\tau_{xz} = 0$

At z = +d, $\tau_{xz} = \left(\frac{6Fd}{d^2}\right) - \left(\frac{6F}{d^3}\right)d^2 = 0$

At z = -d, $\tau_{xz} = -\left(\frac{6F}{d^2}\right)d - \left(\frac{6F}{d^3}\right)(-d^2) = -\left(\frac{12F}{d}\right)$

The variation of τ_{xz} is shown in figure below.

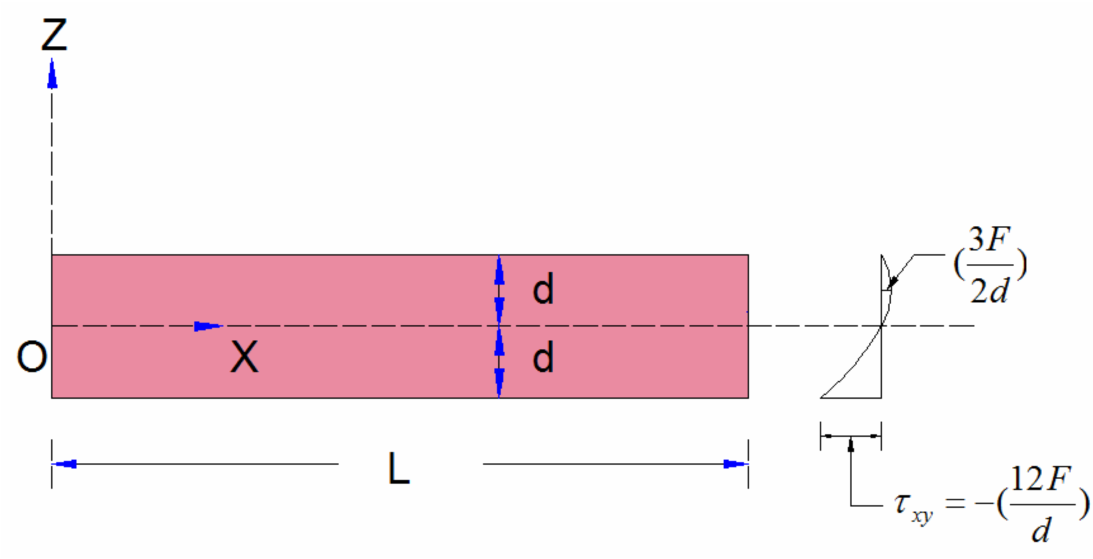


Figure 5.13 Variation of τ_{xz}

Example 5.4

Investigate what problem of plane stress is satisfied by the stress function

$$\varphi = \frac{3F}{4d}\left[xy - \frac{xy^3}{3d^2}\right] + \frac{p}{2}y^2$$

applied to the region included in y = 0, y = d, *x* = 0 on the side *x* positive.

Solution: The given stress function may be written as

$$\varphi = \left(\frac{3F}{4d}\right)xy - \left(\frac{1}{4}\frac{Fxy^3}{d^3}\right) + \left(\frac{p}{2}\right)y^2$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} = 0$$

$$\frac{\partial^2\varphi}{\partial y^2} = -\left(\frac{3\times 2}{4}.\frac{Fxy}{d^3}\right) + \frac{2p}{2} = p - \left(1.5\frac{F}{d^3}\right)xy$$

and $\dfrac{\partial^2\phi}{\partial x\partial y} = \dfrac{3F}{4d} - \dfrac{3}{4}\dfrac{Fy^2}{d^3}$

Hence the stress components are

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} = p - 1.5\frac{F}{d^3}xy$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \frac{3}{4}\frac{Fy^2}{d^3} - \frac{3F}{4d}$$

(a) Variation of σ_x

$$\sigma_x = p - \left(1.5\frac{F}{d^3}\right)xy$$

When x = 0 and y = 0 or $\pm d$, $\sigma_x = p$ (i.e., constant across the section)

When $x = L$ and y = 0, $\sigma_x = p$

When $x = L$ and y = +d, $\sigma_x = p - \left(1.5\frac{FL}{d^2}\right)$

When $x = L$ and y = -d, $\sigma_x = P + 1.5\frac{FL}{d^2}$

Thus, at $x = L$, the variation of σ_x is linear with y.

The variation of σ_x is shown in the figure below.

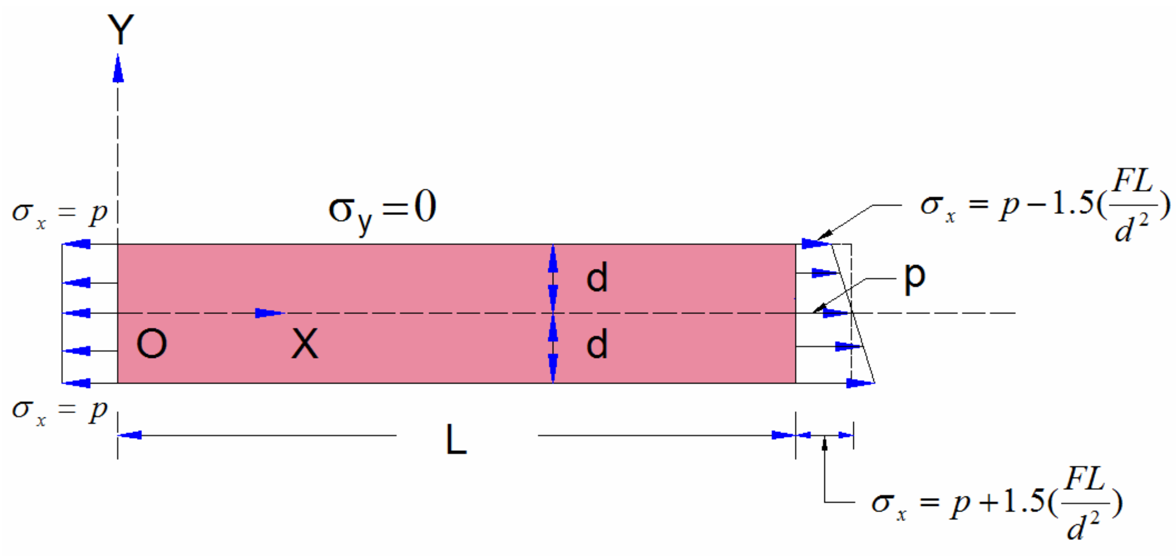


Figure 5.14 Variation of stress σ_x

(b) Variation of σ_z

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$\therefore \sigma_y$ is zero for all value of x and y

(c) Variation of τ_{xy}

$$\tau_{xy} = \left(\frac{3}{4}\frac{Fy^2}{d^3}\right) - \left(\frac{3}{4}\frac{F}{d}\right)$$

Thus, τ_{xy} varies parabolically with z. However, it is independent of x, i.e., it's value is the same for all values of x.

$\therefore$ At $y = 0$, $\tau_{xy} = -\left(\frac{3}{4}\frac{F}{d}\right)$

At $y = \pm d$, $\tau_{xy} = \left[\frac{3}{4}\frac{F}{d^3}(d)^2\right] - \left[\frac{3}{4}\frac{F}{d}\right] = 0$

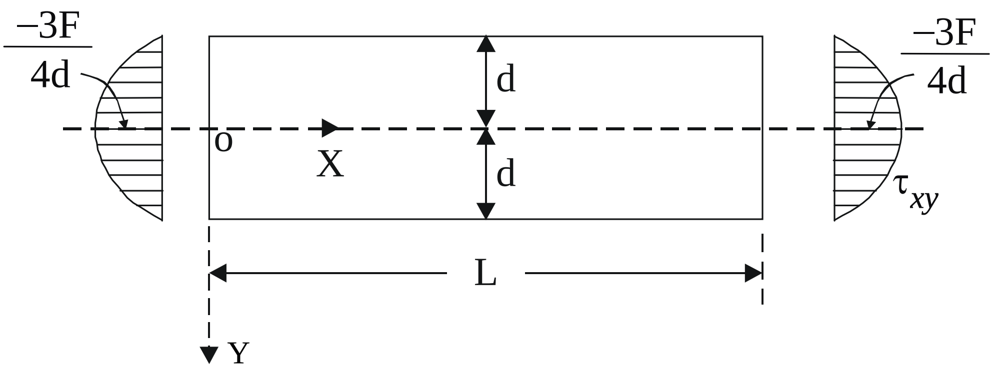


Figure 5.15 Variation of shear stress τ_{xy}

The stress function therefore solves the problem of a cantilever beam subjected to point load F at its free end along with an axial stress of p.

Example 5.5

Show that the following stress function satisfies the boundary condition in a beam of rectangular cross-section of width 2*h* and depth *d* under a total shear force *W*.

$$\phi = -\left[\frac{W}{2hd^3}xy^2(3d - 2y)\right]$$

***Solution*:** $\sigma_x = \frac{\partial^2\phi}{\partial y^2}$

Now, $\frac{\partial \phi}{\partial y} = -\frac{W}{2hd^3}\left[6xyd - 6xy^2\right]$

$$\frac{\partial^2 \phi}{\partial y^2} = -\frac{W}{2hd^3}\left[6xd - 12xy\right]$$

$$\therefore \sigma_x = -\frac{W}{hd^3}\left[3xd - 6xy\right]$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$$

and $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

$$= \frac{W}{2hd^3}\left[6yd - 6y^2\right]$$

$$= \frac{W}{hd^3}\left[3yd - 3y^2\right]$$

Also, $\nabla^4 \phi = \left[\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{2\partial^4}{\partial x^2 \partial y^2}\right]\phi = 0$

Boundary conditions are

(a) $\sigma_y = 0 \ \ for \ \ y = 0 \ and \ d$

(b) $\tau_{xy} = 0 \ \ for \ \ y = 0 \ \ and \ \ d$

(c) $\int_0^d \tau_{xy}.2h.dy = W \ \ for \ \ x = 0 \ and \ L$

(d) $M = \int_0^d \sigma_x.2h.dy = 0 \ \ for \ \ x = 0 \ and \ x = L, \ M = WL$

(e) $\int_0^d \sigma_x.2h.y.dy = 0 \ \ for \ \ x = 0 \ and \ x = L$

Now, Condition (a)

This condition is satisfied since $\sigma_y = 0$

Condition (b)

$$\frac{W}{hd^3}\left[3d^2 - 3d^2\right] = 0$$

Hence satisfied.

Condition (c)

$$\int_0^d \frac{W}{hd^3}\left[3yd - 3y^2\right]2hdy$$

$$= \int_0^d \frac{2W}{d^3}\left[3yd - 3y^2\right]dy$$

$$= \frac{2W}{d^3}\left[\frac{3y^2 d}{2} - y^3\right]_0^d$$

$$= \frac{2W}{d^3}\left[\frac{3d^3}{2} - d^3\right]$$

$$= \frac{2W}{d^3}.\frac{d^3}{2}$$

$= W$

Hence satisfied.

Condition (d)

$$\int_0^d -\frac{W}{hd^3}\left[3xd - 6xy\right]2hdy$$

$$= -\frac{2W}{d^3}\left[3xyd - 3xy^2\right]_0^d$$

$= 0$

Hence satisfied.

Condition (e)

$$\int_0^d -\frac{W}{hd^3}\left[3xd - 6xy\right]2h.ydy$$

$$= -\frac{2W}{d^3}\left[\frac{3xdy^2}{2} - 2xy^3\right]_0^d$$

$$= -\frac{2W}{d^3}\left[\frac{3xd^3}{2} - 2xd^3\right]$$

$$= -\frac{2W}{d^3}\left[-\frac{1}{2}xd^3\right]$$

$$= Wx$$

Hence satisfied